Basin attractors for various methods

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Abstract

There are many methods for the solution of a nonlinear algebraic equation. The methods are classified by the order, informational efficieny and efficiency index. Here we consider another criterion, namely the basin of attraction of the method and its dependence on the order. We discuss several methods of various orders and present the basin of attraction for several examples. It can be seen that ....

Keywords:

1 Introduction

There is a vast literature for the numerical solution of nonlinear equations. In general methods are classified as bracketing, fixed point or hybrid. In the first class one starts with an initial interval in which the function changes sign and at each iteration step the interval shrinks. In the fixed point methods one starts with an initial point and create a sequence that should converge to the desired solution. The methods are also classified by their order of convergence, p, and the number of function- (and derivative-) evaluation per step, denoted by d. There are two efficiency measures defined as $I = p/d$ (informational efficiency) and $E = p^{1/d}$ (efficiency index). Methods for the approximation of multiple roots are also available in the literature. Some of these methods require the knowledge of the multiplicity in advance.

Here we consider another criterion for comparison. The idea of basin of attraction of some root-finding methods was introduced by Stewart [13]. He compared Newton's method to the third order methods given by Halley [2], Chebyshev [3] and Laguerre [4]. In an ideal case, if a function has n distinct zeros, then the plane is divided to n basins. For example, if we have the polynomial $z^3 - 1$, then the roots are $z = 1$ and $z = \frac{-1 \pm \sqrt{3}i}{2}$, see figure 1. Ideally the basins boundaries are straight lines. Actually, depending on the numerical method, we find the basin boundaries are much more complex, see example 5 later.

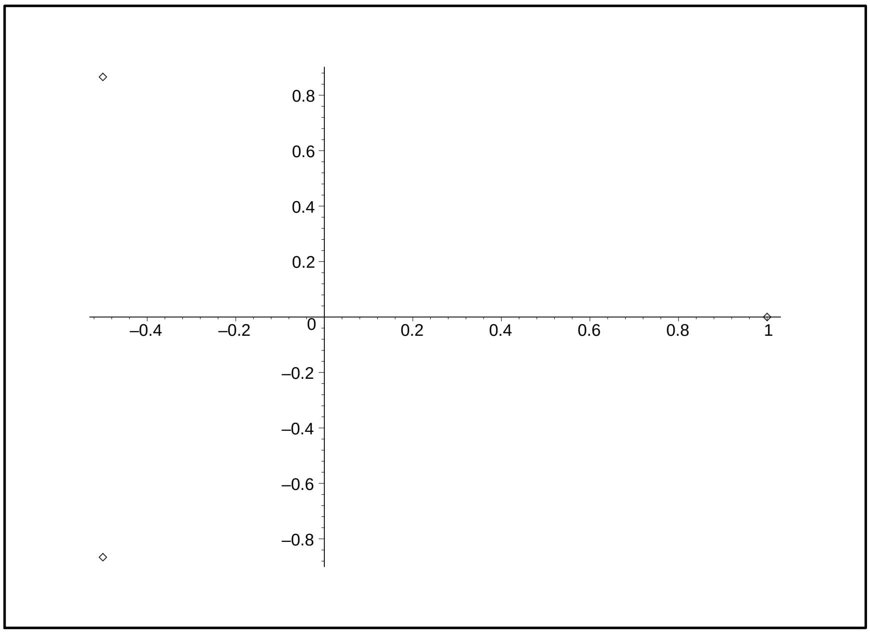


Figure 1: Location of the roots of $z^3 - 1$

Our study considers seven (????) methods of increasing order, two of which were considered by Stewart. We include optimal methods of order $p = 2, 4, 8, 16$. Note that a method of order $p = 2^n$ is optimal (see [6]) in the sense that it requires $n + 1$ function- (and derivative-) evaluations per cycle. The methods we consider here with their order of convergence are:

1. Newton's method ($p = 2$)

2. Halley's method ($p = 3$)
3. King's optimal method ($p = 4$)
4. Kung-Traub's optimal method ($p = 4$)
5. Murakami's method ($p = 5$)
6. Neta's method ($p = 6$)
7. Neta-Chun's method ($p = 6$)
8. Neta-Johnson's method ($p = 8$)
9. Neta's optimal method ($p = 8$)
10. Neta's optimal method ($p = 16$)

The reason why we introduced more than one optimal fourth order method and more than one sixth order method will be clarified later.

Newton's optimal method (see e.g. Conte and deBoor [1]) which is of second order for simple roots and given by

$$x_{n+1} = x_n - \frac{f_n}{f'_n} \tag{1}$$

where $f_n = f(x_n)$ and similarly for the derivative. Halley's method [2] is of third order and given by

$$x_{n+1} = x_n - \frac{\dfrac{f_n}{f'_n}}{1 - \dfrac{f''_n}{2f'_n}\dfrac{f_n}{f'_n}}. \tag{2}$$

King's fourth order optimal family of methods [5] is given by

$$\begin{aligned} w_n &= x_n - \frac{f_n}{f'_n} \\ x_{n+1} &= w_n - \frac{f(w_n)}{f'_n}\frac{f_n + \beta f(w_n)}{f_n + (\beta - 2)f(w_n)}. \end{aligned} \tag{3}$$

Another optimal fourth order method is due to Kung and Traub [6] given by

$$\begin{aligned} w_n &= x_n - \frac{f_n}{f'_n} \\ x_{n+1} &= w_n - \frac{f(w_n)}{f'_n}\frac{1}{[1 - f(w_n)/f_n]^2}. \end{aligned} \tag{4}$$

Murakami's fifth order method [7] is given by

$$x_{n+1} = x_n - a_1 u_n - a_2 w_2(x_n) - a_3 w_3(x_n) - \psi(x_n), \tag{5}$$

where

$$\begin{aligned} w_2(x_n) &= \frac{f_n}{f'(x_n - u_n)}, \\ w_3(x_n) &= \frac{f_n}{f'(x_n + \beta u_n + \gamma w_2(x_n))}, \\ \psi(x_n) &= \frac{f_n}{b_1 f'_n + b_2 f'(x_n - u_n)}. \end{aligned} \tag{6}$$

To get fifth order, Murakami suggested several possibilities and we picked the following

$$\begin{aligned} &\gamma = 0, \qquad a_1 = .3, \qquad a_2 = -.5, \qquad a_3 = \frac{2}{3}, \\ &b_1 = -\frac{15}{32}, \qquad b_2 = \frac{75}{32}, \qquad \beta = -\frac{1}{2} \end{aligned} \tag{7}$$

Neta's sixth order method [8] is given by

$$\begin{aligned} w_n &= x_n - \frac{f_n}{f'_n}, \\ z_n &= w_n - \frac{f(w_n)}{f'_n}\frac{f_n + \beta f(w_n)}{f_n + (\beta - 2) f(w_n)}, \\ x_{n+1} &= z_n - \frac{f(z_n)}{f'_n}\frac{f_n - f(w_n)}{f_n - 3 f(w_n)}. \end{aligned} \tag{8}$$

Note that the first two steps are King's method.

Another sixth order method due to Neta and Chun is based on the first steps of Kung and Traub,

$$
\begin{aligned}
w_n &= x_n - \frac{f_n}{f'_n} \\
z_n &= w_n - \frac{f(w_n)}{f'_n} \frac{1}{[1 - f(w_n)/f_n]^2} \\
x_{n+1} &= z_n - \frac{f(z_n)}{f'_n} \frac{1}{[1 - f(w_n)/f_n - f(z_n)/f_n]^2}.
\end{aligned}
\tag{9}
$$

Neta and Johnson [9] have developed an eighth order method based on Jarratt's (method [10]

$$
\begin{aligned}
z_n &= x_n - \frac{f(x_n)}{\frac{1}{6}f'(x_n) + \frac{1}{6}f'(y_n) + \frac{2}{3}f'(\eta_n)} \\
x_{n+1} &= z_n - \frac{f(z_n)}{f'(x_n)} \frac{f'(x_n) + f'(y_n) + a_2 f'(\eta_n)}{(-1 - a_2)f'(x_n) + (3 + a_2)f'(y_n) + a_2 f'(\eta_n)}
\end{aligned}
\tag{10}
$$

where

$$
\begin{aligned}
u_n &= \frac{f(x_n)}{f'(x_n)} \\
y_n &= x_n - u_n \\
v_n &= \frac{f(x_n)}{f'(y_n)} \\
\eta_n &= x_n - \frac{1}{8}u_n - \frac{3}{8}v_n
\end{aligned}
\tag{11}
$$

In our experiments we have used $a_2 = -1$. This is not an optimal method since it requires 2 function- and 3 derivative-evaluation per cycle.

Another eighth order method is the optimal method due to Neta and Petković [11]. It is based on Kung and Traub's optimal fourth order method [6].

$$
\begin{aligned}
w_n &= x_n - \frac{f_n}{f'_n} \\
z_n &= x_n - \frac{f(w_n)}{f'_n}\frac{1}{[1 - f(w_n)/f_n]^2} \\
x_{n+1} &= x_n - \frac{f_n}{f'_n} + c_n f_n^2 - d_n f_n^3
\end{aligned}
\tag{12}
$$

where

$$
\begin{aligned}
d_n &= \frac{1}{[f(w_n) - f(x_n)]\,[f(w_n) - f(z_n)]}\left[\frac{w_n - x_n}{f(w_n) - f(x_n)} - \frac{1}{f'_n}\right] \\
&- \frac{1}{[f(w_n) - f(z_n)]\,[f(z_n) - f(x_n)]}\left[\frac{z_n - x_n}{f(z_n) - f(x_n)} - \frac{1}{f'_n}\right] \\
c_n &= \frac{1}{f(w_n) - f(x_n)}\left[\frac{w_n - x_n}{f(w_n) - f(x_n)} - \frac{1}{f'_n}\right] - d_n\,[f(w_n) - f(x_n)]
\end{aligned}
\tag{13}
$$

Neta's sixteenth order optimal method [12] is given by

$$
\begin{aligned}
w_n &= x_n - \frac{f_n}{f'_n}, \\
z_n &= w_n - \frac{f(w_n)}{f'_n}\frac{f_n + \beta f(w_n)}{f_n + (\beta - 2) f(w_n)}, \\
t_n &= x_n - \frac{f_n}{f'_n} + c_n f_n^2 - d_n f_n^3 \\
x_{n+1} &= x_n - \frac{f_n}{f'_n} + \rho_n f_n^2 - \gamma_n f_n^3 + q_n f_n^4
\end{aligned}
\tag{14}
$$

where c_n and d_n are given by (13) and

$$\begin{aligned} q_n &= \frac{\dfrac{\phi(t_n)-\phi(z_n)}{F(t_n)-F(z_n)} - \dfrac{\phi(w_n)-\phi(z_n)}{F(w_n)-F(z_n)}}{F(t_n)-F(w_n)}, \\ \gamma_n &= \frac{\phi(t_n)-\phi(z_n)}{F(t_n)-F(z_n)} - q_n\left(F(t_n)+F(z_n)\right), \\ \rho_n &= \phi(t_n) - \gamma_n F(t_n) - q_n F^2(t_n), \end{aligned} \tag{15}$$

and for $\delta_n = w_n, z_n, t_n$

$$\begin{aligned} F(\delta_n) &= f(\delta_n) - f_n, \\ \phi(\delta_n) &= \frac{(\delta_n - x_n)}{F^2(\delta_n)} - \frac{1}{f'_n F(\delta_n)}. \end{aligned} \tag{16}$$

2 numerical experiments

We have used the above methods for 14 different polynomials. Some are real and some are complex polynomials. Four examples have only real roots and the rest have a combination of real and complex ones. All the roots are simple. In the first case we have taken the cubic polynomial

$$x^3 + 4x^2 - 10 \tag{17}$$

Clearly, one root is real and the other two are complex conjugate.

Note that the basin of attraction of each root is larger for Halley's method than Newton's. We have shown the results for King's method using the parameter $\beta = -1/2$. The results are not much better for several other values of β we tried. The basins of attraction for the optimal Kung-Traub method is better than any of the King's method (notice the second quadrant). Therefore, we will not use King's method in the rest of the experiments. Murakami's fifth order method gives basins of attraction similar to Newton's. On the other hand, Neta's sixth order method which is based on King's method and uses $\beta = -1/2$ shows some chaotic behavior in the second

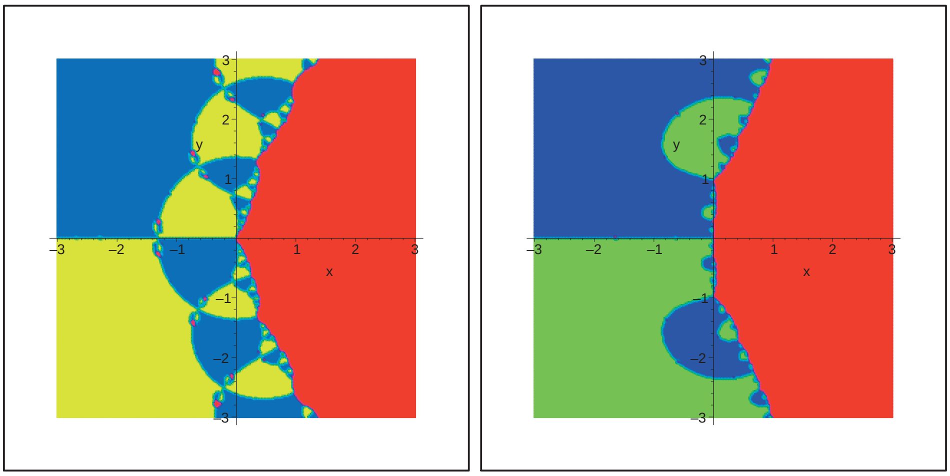


Figure 2: Newton's (left) and Halley's method (right) for the cubic polynomial whose roots are: 1.365230013, -2.682615007+.3582593602i, -2.682615007-.3582593602i

quadrant. One sees too many points there that converge to the root in the third quadrant. This is similar to the results for King's method.

Neta-Johnson's eighth order method has basins of attraction similar to Newton's method. On the other hand, the optimal eighth order method (12) is much more chaotic. The 16th order optimal method has a smaller basin of attraction for the real root, but it doesn't show the chaotic behavior of King's, Neta's sixth order and Neta-Petkovic eighth order. In general, one cannot say that increasing the order of the method will adversly affect the basins of attraction very much.

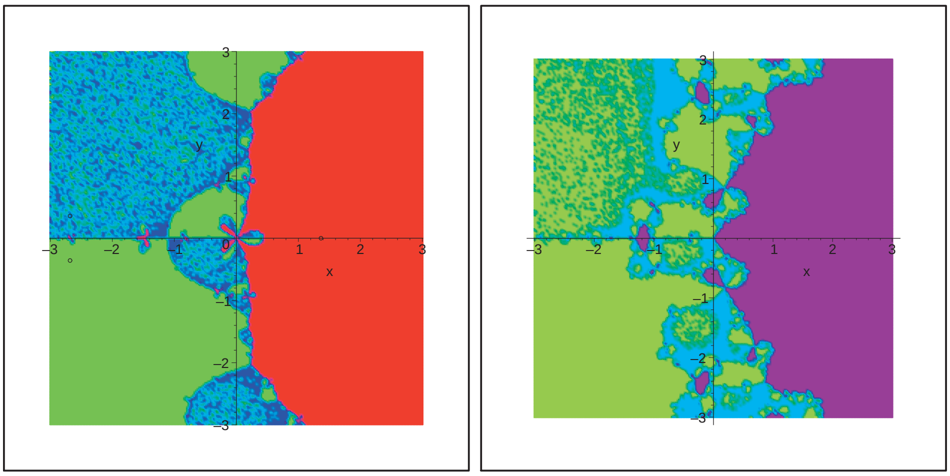


Figure 3: King's fourth order method with $\beta = -1/2$ (left) and Kung and Traub's fourth order method (right)

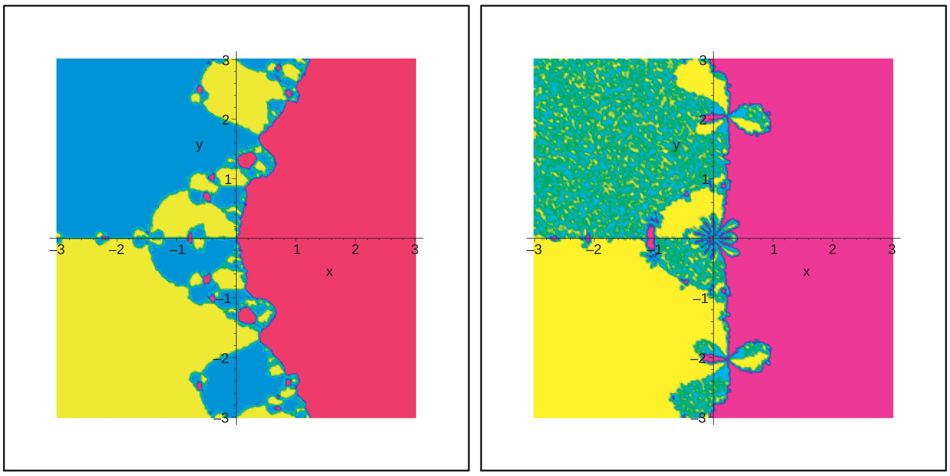


Figure 4: Murakami's fifth order method (left) and Neta's sixth order method with $\beta = -1/2$ (right)

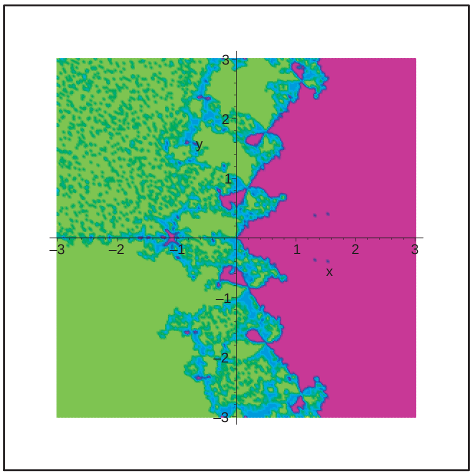


Figure 5: Neta's sixth order method based on Kung-Traub scheme

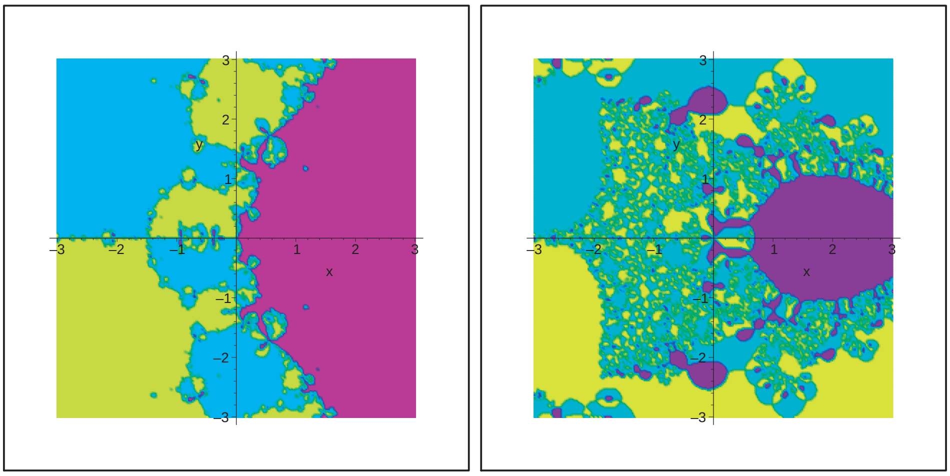


Figure 6: Neta and Johnson's eighth order (left) and Neta and Petkovic's optimal eighth order method (right)

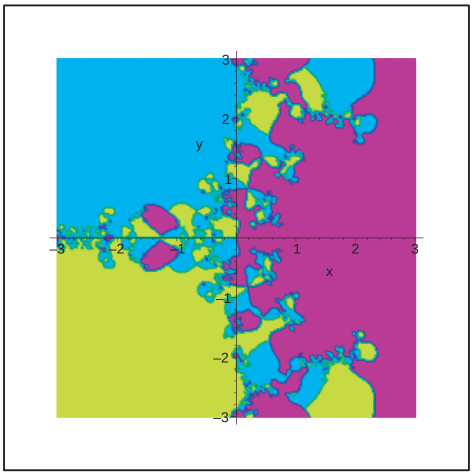


Figure 7: Neta's sixteenth order method

In our next example, we have taken the real quartic polynomial whose roots are $-25, -1, 20, 25$.

$$x^4 - 19x^3 - 645x^2 + 11875x + 12500 \tag{18}$$

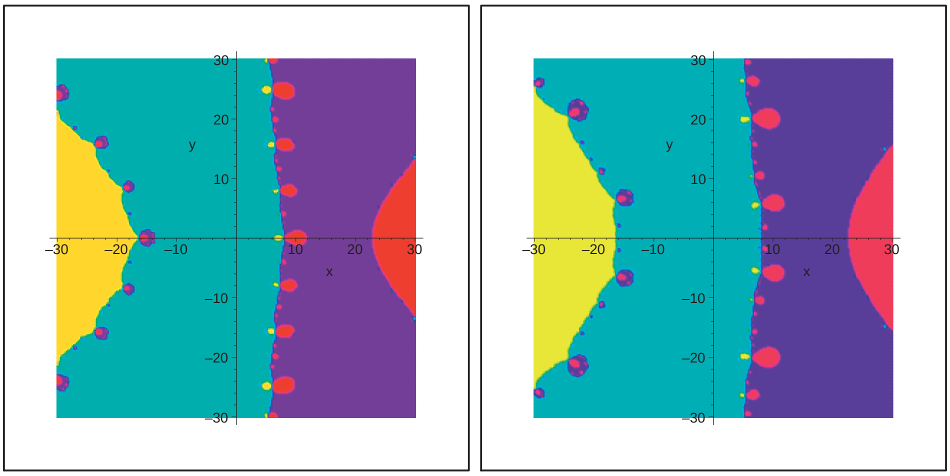


Figure 8: Newton's (left) and Halley's method (right) for the quartic polynomial whose roots are: -25, -1, 20, 25

As can be seen, the optimal eighth and sixteenth order methods as well as the sixth order are not doing very well.

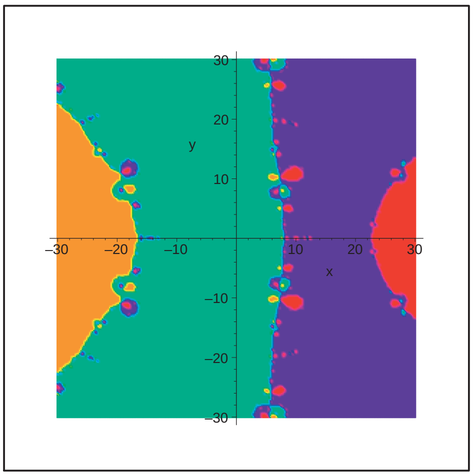


Figure 9: Kung and Traub's fourth order method (left) and Murakami's fifth order method (right)

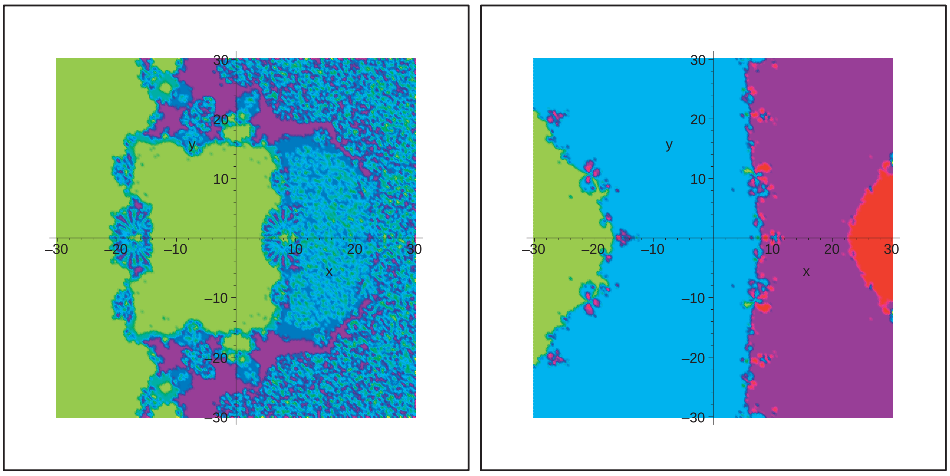


Figure 10: Neta's sixth order method (left) and Neta and Johnson's eighth order method (right)

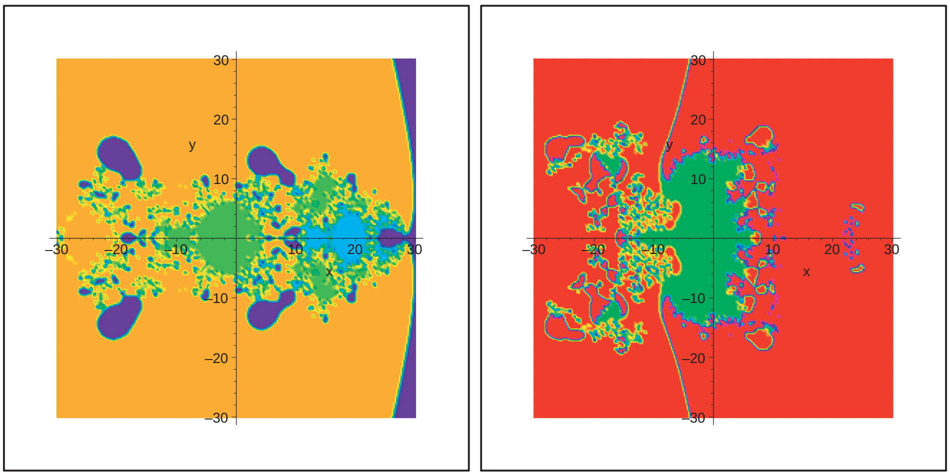


Figure 11: Neta and Petkovic's optimal eighth order (left) and Neta's sixteenth order method (right)

In our next example, we have taken a cubic polynomial with real simple roots.

$$x^3 - x \tag{19}$$

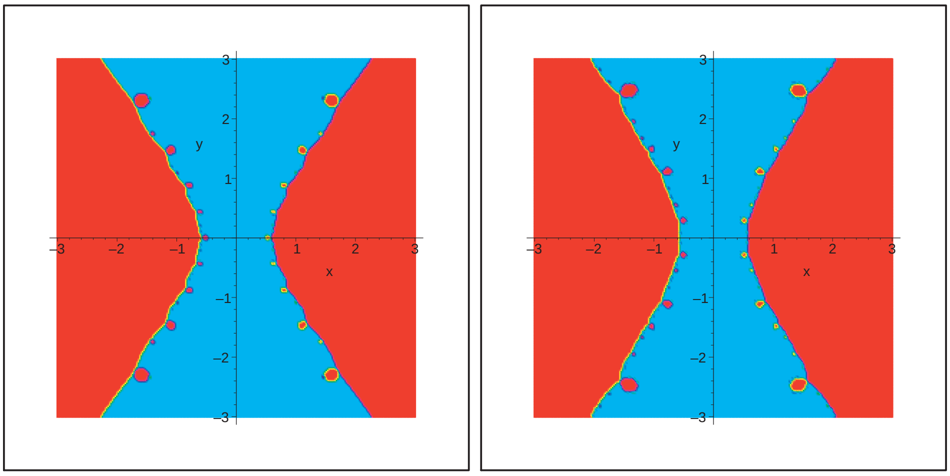


Figure 12: Newton's (left) and Halley's method (right) for the cubic polynomial whose roots are: -1, 0, 1

As can be seen, the optimal eighth and sixteenth order methods as well as the sixth order are not doing very well.

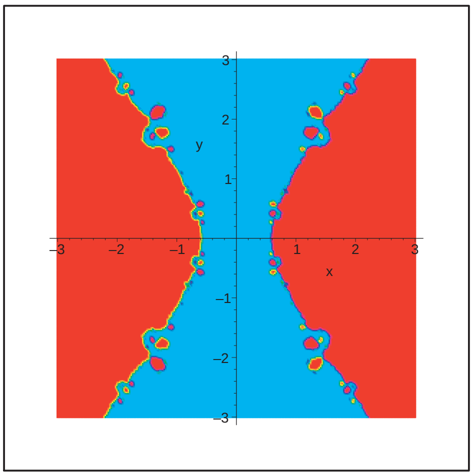


Figure 13: Kung and Traub's fourth order method (left) and Murakami's fifth order method (right)

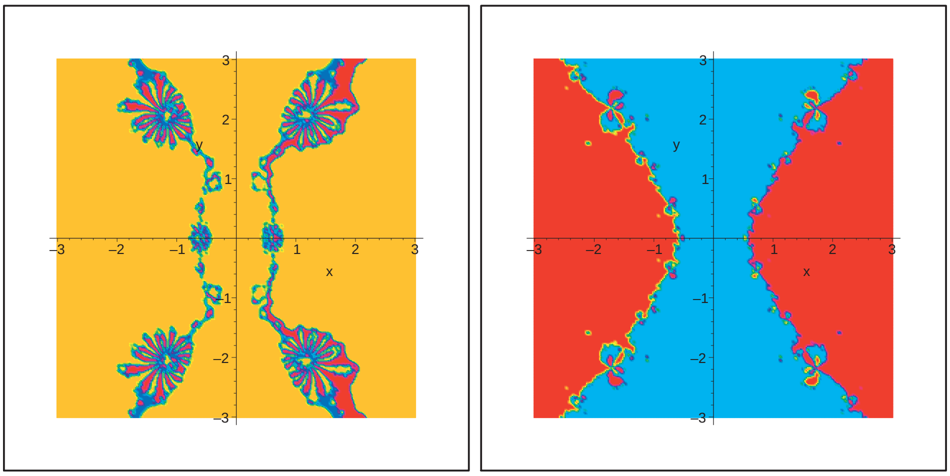


Figure 14: Neta's sixth order method (left) and Neta and Johnson's eighth order method (right)

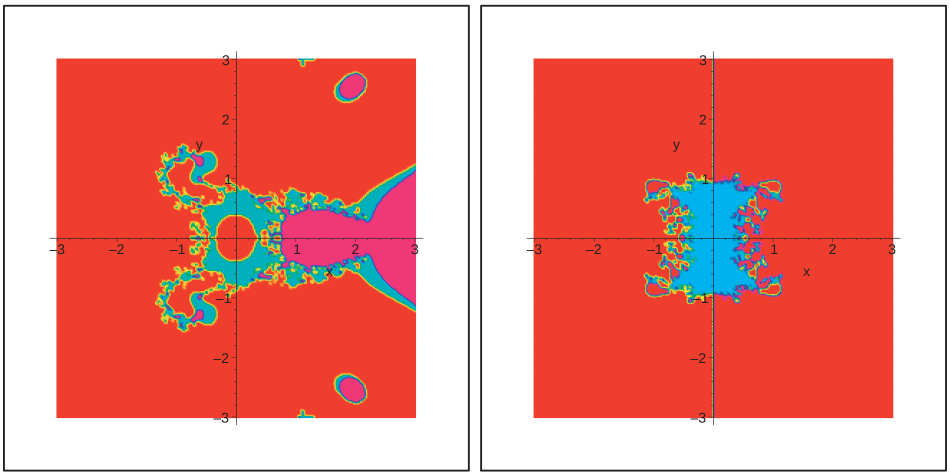


Figure 15: Neta and Petkovic's optimal eighth order (left) and Neta's sixteenth order method (right)

In our next exaple, we took a quintic polynomial with real simple roots.

$$x^5 - 5x^3 + 4x \tag{20}$$

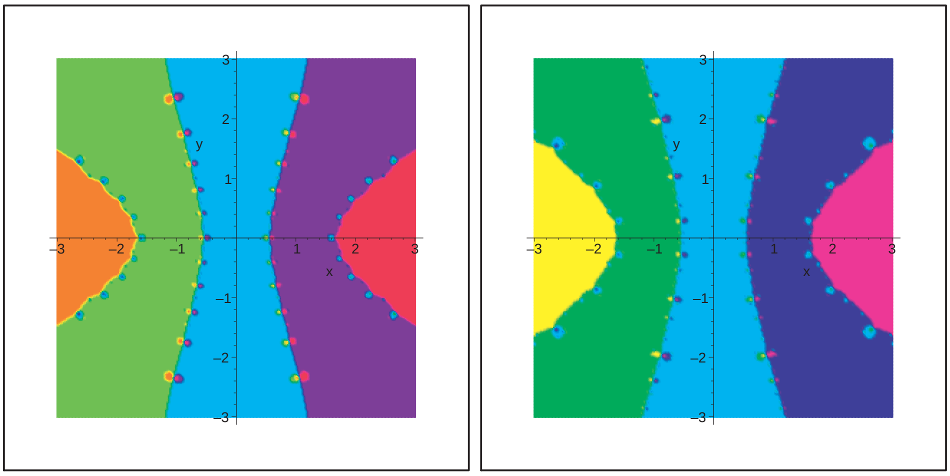


Figure 16: Newton's (left) and Halley's method (right) for the quintic polynomial whose roots are: -2, -1, 0, 1, 2

As can be seen, the optimal eighth and sixteenth order methods as well as the sixth order are not doing very well.

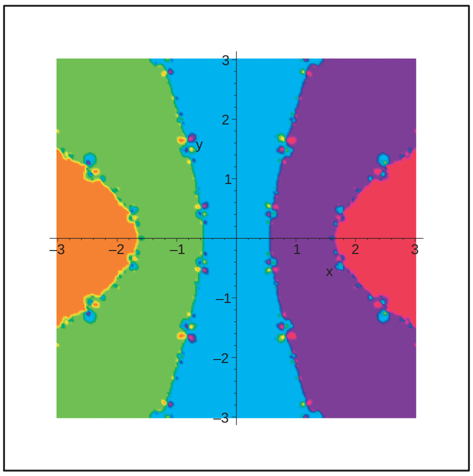


Figure 17: Kung and Traub's fourth order method (left) and Murakami's fifth order method (right)

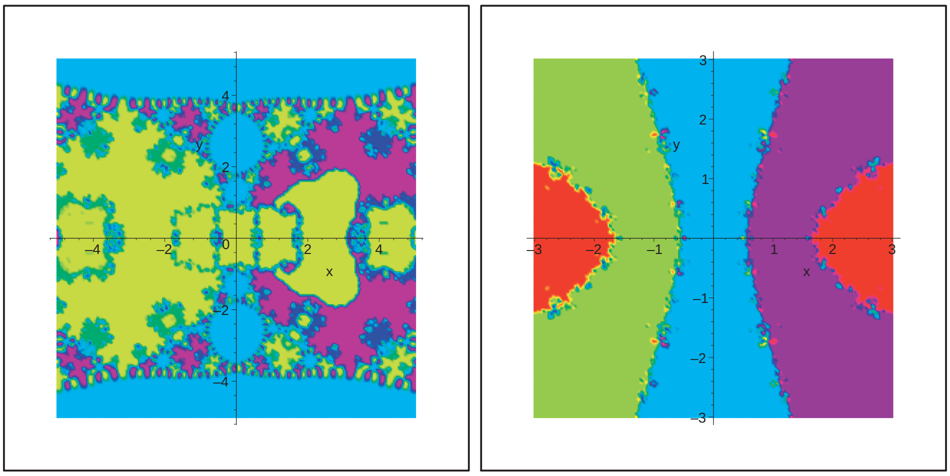


Figure 18: Neta's sixth order method (left) and Neta and Johnson's eighth order method (right)

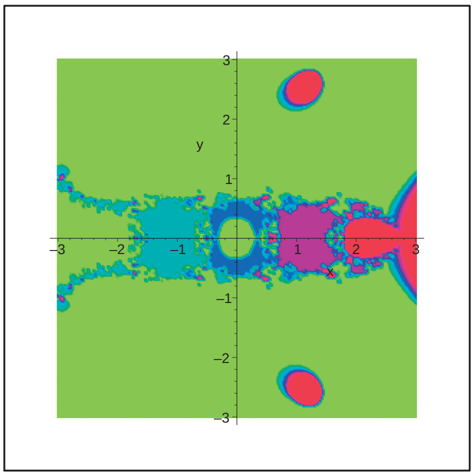


Figure 19: Neta and Petkovic's optimal eighth order (left) and Neta's sixteenth order method (right)

In the next three examples we have taken polynomial yielding the roots of unity. The first is a cubic then quintic and lastly a polynomial of degree 7.

$$x^3 - 1 \tag{21}$$

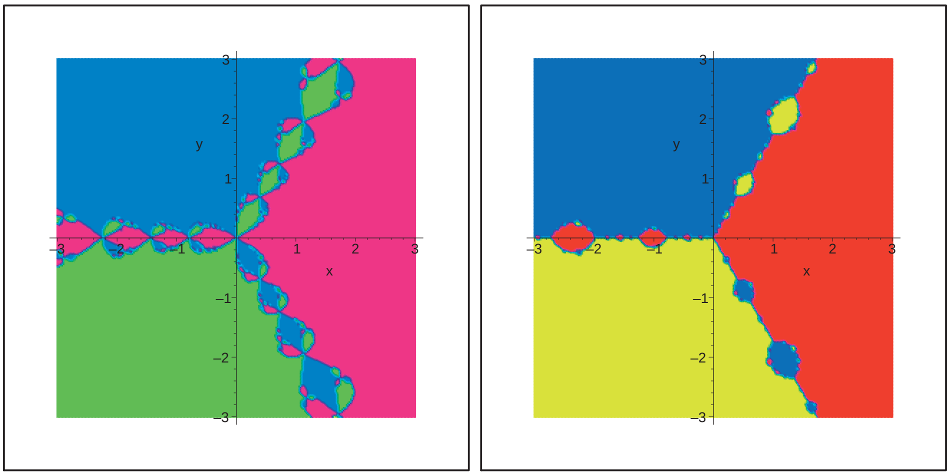


Figure 20: Newton's (left) and Halley's method (right) for the cubic polynomial whose roots are: 1., -.5+.8660254040i, -.5-.8660254040i

As can be seen, the optimal eighth order and the sixth order methods are not doing very well.

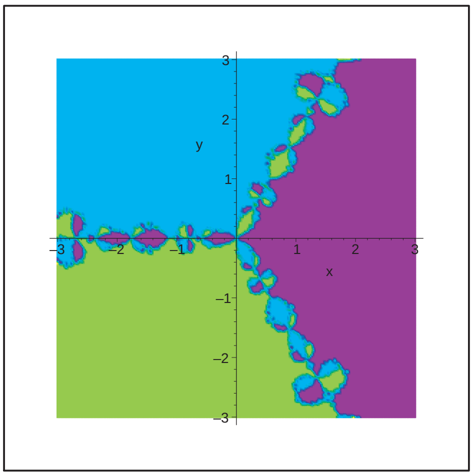


Figure 21: Kung and Traub's fourth order method (left) and Murakami's fifth order method (right)

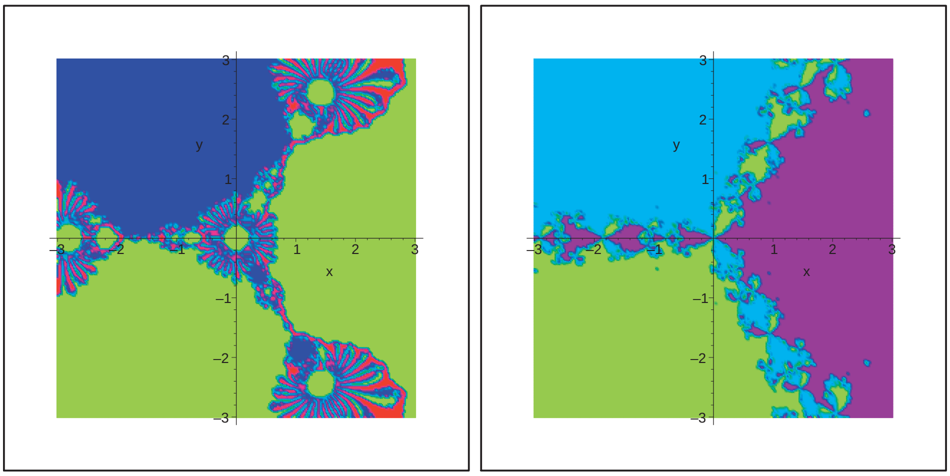


Figure 22: Neta's sixth order method (left) and Neta and Johnson's eighth order method (right)

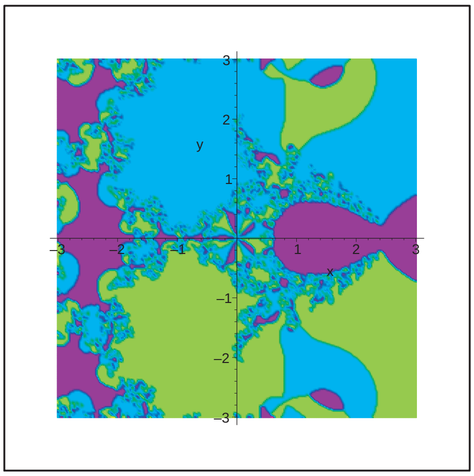


Figure 23: Neta and Petkovic's optimal eighth order (left) and Neta's sixteenth order method (right)

$$x^5 - 1 \tag{22}$$

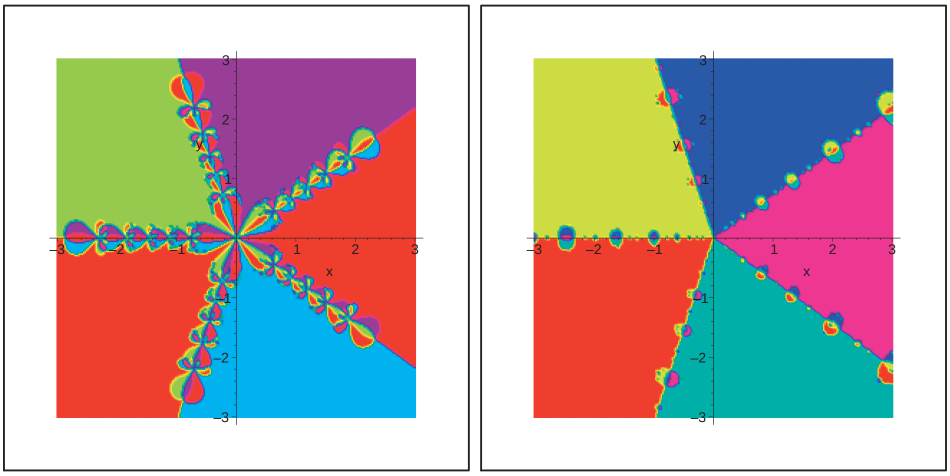


Figure 24: Newton's (left) and Halley's method (right) for the five roots of unity

As can be seen, the optimal eighth order and the sixth order methods are not doing very well.

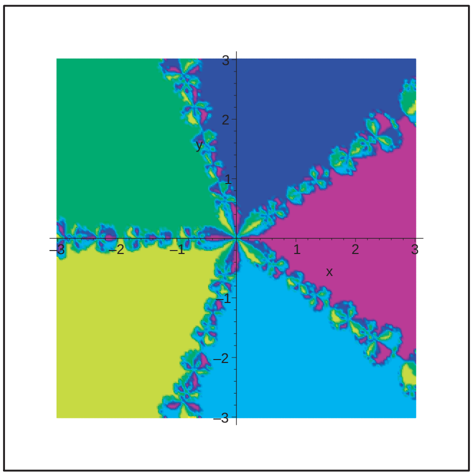


Figure 25: Kung and Traub's fourth order method (left) and Murakami's fifth order method (right)

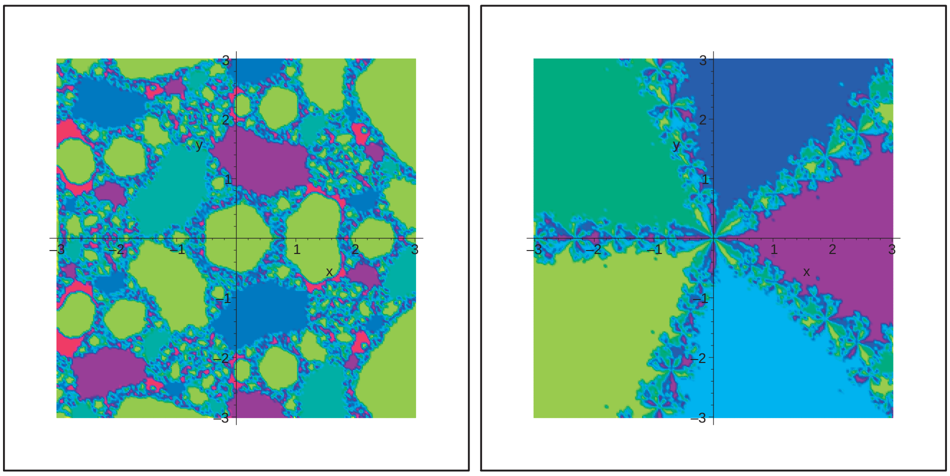


Figure 26: Neta's sixth order method (left) and Neta and Johnson's eighth order method (right)

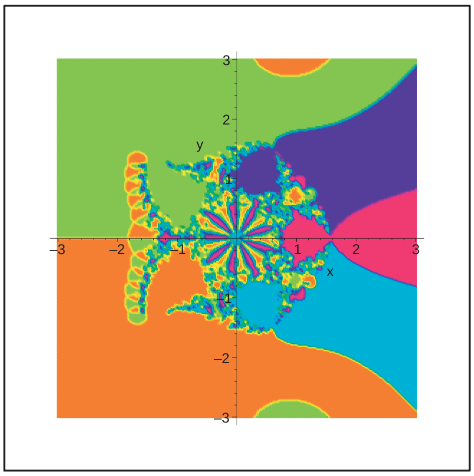


Figure 27: Neta and Petkovic's optimal eighth order (left) and Neta's sixteenth order method (right)

$$x^7 - 1 \tag{23}$$

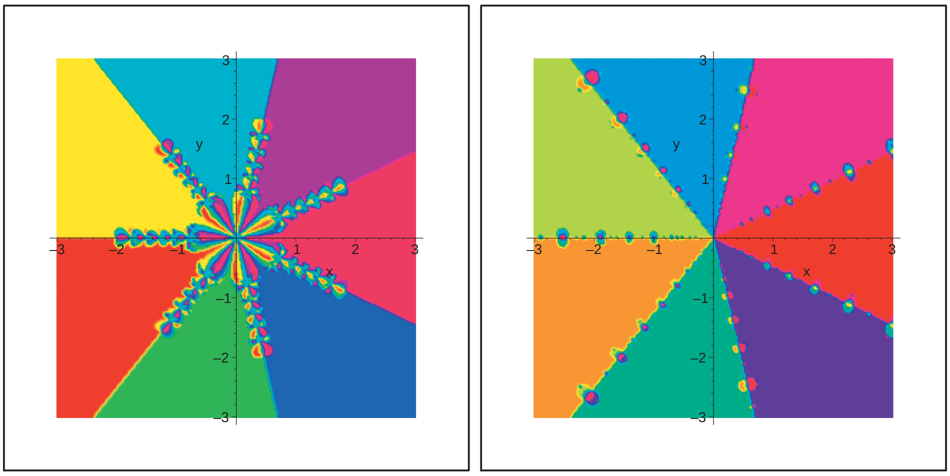


Figure 28: Newton's (left) and Halley's method (right) for the seven roots of unity

As can be seen, only the optimal eighth order method is not doing very well.

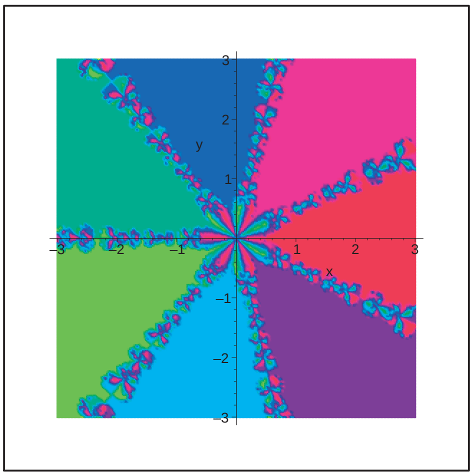


Figure 29: Kung and Traub's fourth order method (left) and Murakami's fifth order method (right)

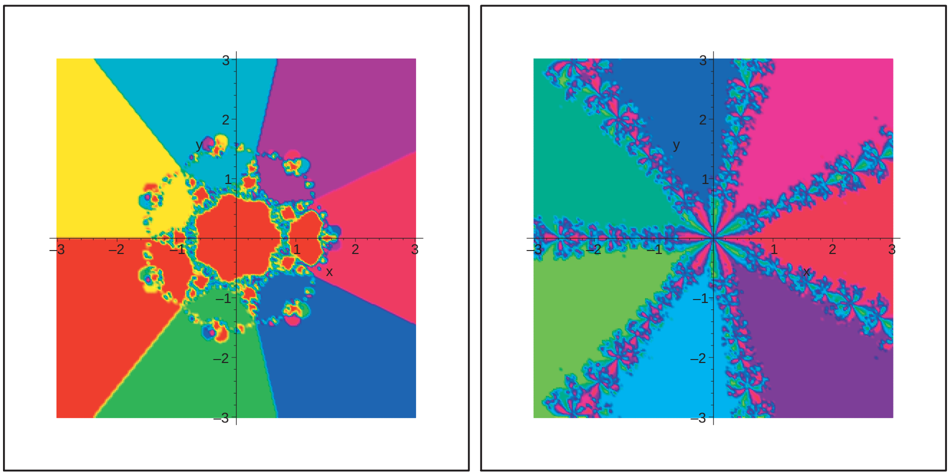


Figure 30: Neta's sixth order method (left) and Neta and Johnson's eighth order method (right)

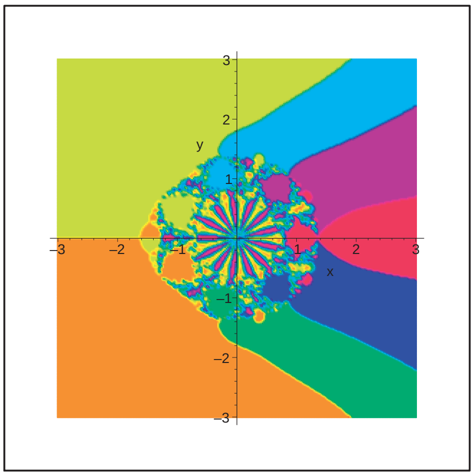


Figure 31: Neta and Petkovic's optimal eighth order (left) and Neta's sixteenth order method (right)

The last 6 examples are using complex polynomials with simple real and complex roots.

$$x^3 + 2x^2 - 3ix^2 - \frac{3}{4}x - \frac{9}{2}ix - \frac{7}{4} - \frac{3}{2}i \tag{24}$$

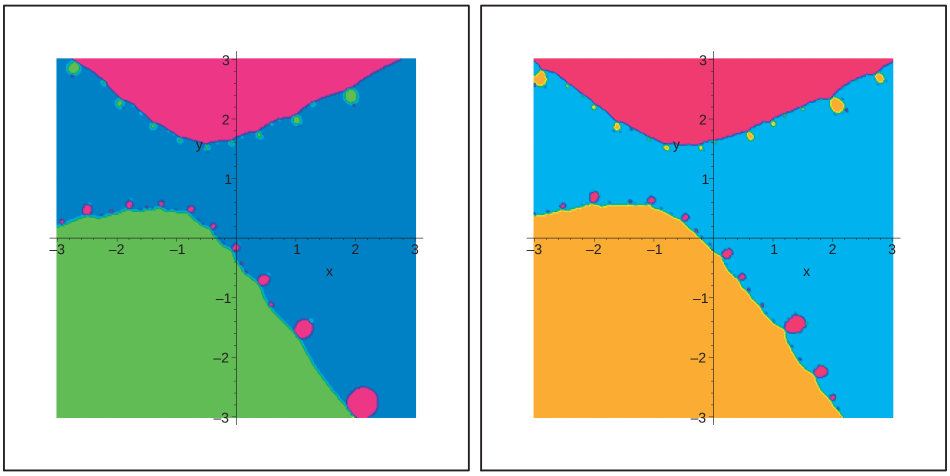


Figure 32: Newton's (left) and Halley's method (right) for the complex cubic polynomial whose roots are: -.5+2i, -.5+i, -1

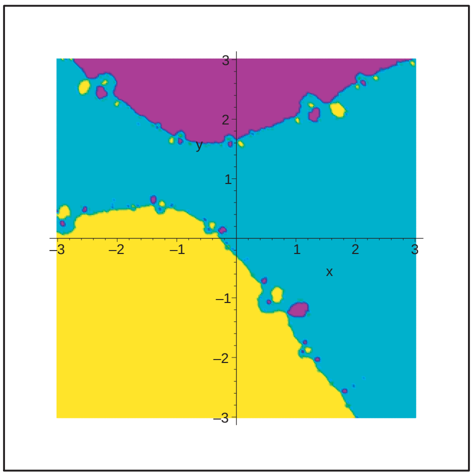


Figure 33: Kung and Traub's fourth order method (left) and Murakami's fifth order method (right)

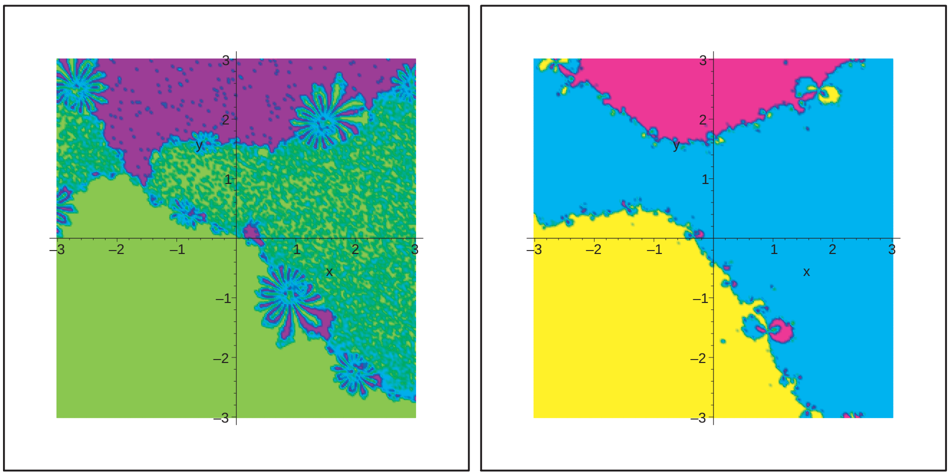


Figure 34: Neta's sixth order method (left) and Neta and Johnson's eighth order method (right)

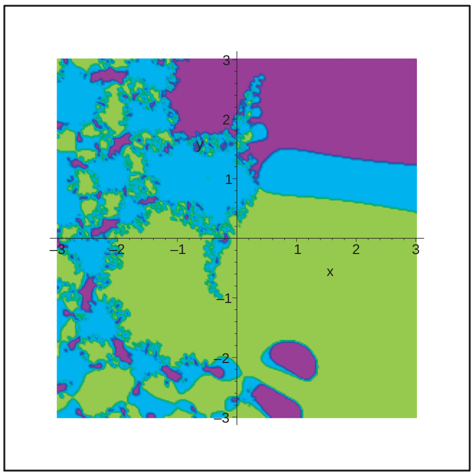


Figure 35: Neta and Petkovic's optimal eighth order (left) and Neta's sixteenth order method (right)

$$x^3 - 3x^2 + 5ix^2 - 51ix - 48x - 54 + 54i \tag{25}$$

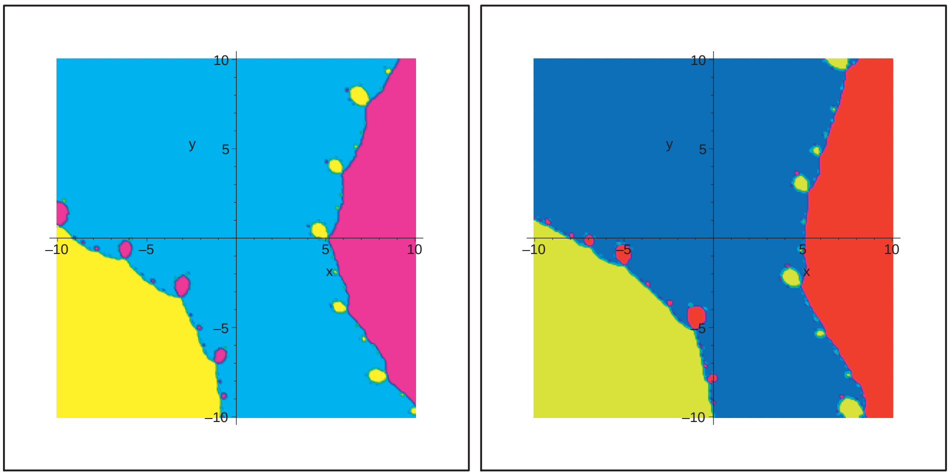


Figure 36: Newton's (left) and Halley's method (right) for the complex cubic polynomial whose roots are: -6-6i, i, 9

As can be seen, the sixth order and the optimal sixteenth order methods are not doing very well.

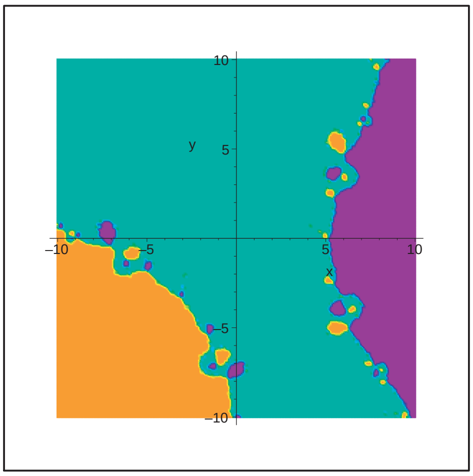


Figure 37: Kung and Traub's fourth order method (left) and Murakami's fifth order method (right)

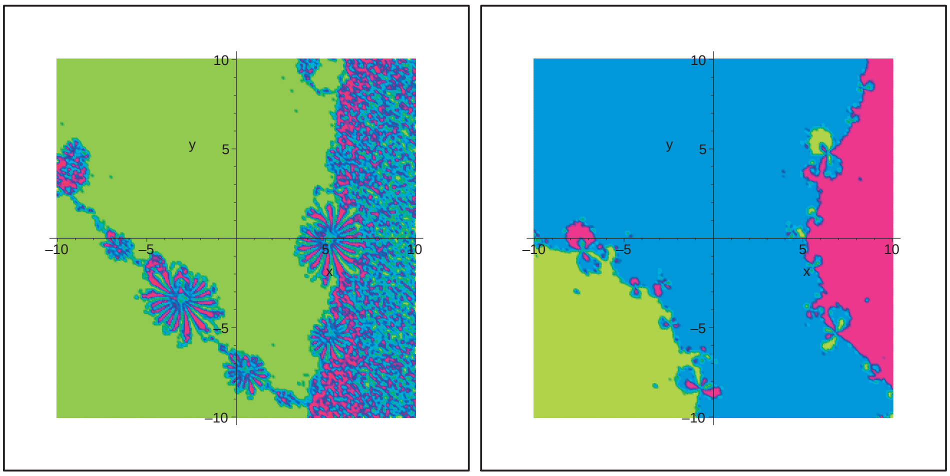


Figure 38: Neta's sixth order method (left) and Neta and Johnson's eighth order method (right)

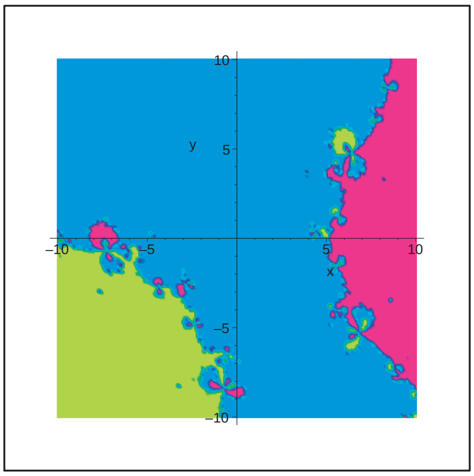


Figure 39: Neta and Petkovic's optimal eighth order (left) and Neta's sixteenth order method (right)

$$x^4 + 20ix^2 - 36 \qquad (26)$$

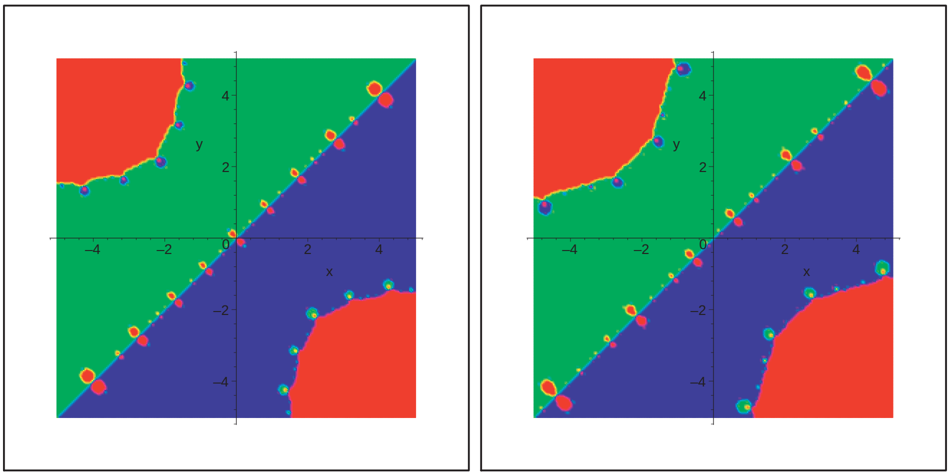


Figure 40: Newton's (left) and Halley's method (right) for the complex quartic polynomial whose roots are: -3+3i, 1-i, -1+i, 3-3i

As can be seen, the optimal eighth and sixteenth order methods as well as the sixth order are not doing very well.

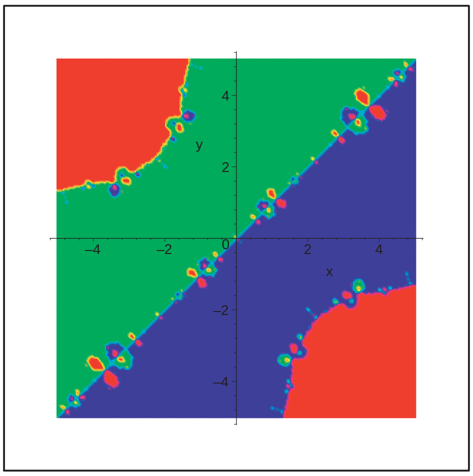


Figure 41: Kung and Traub's fourth order method (left) and Murakami's fifth order method (right)

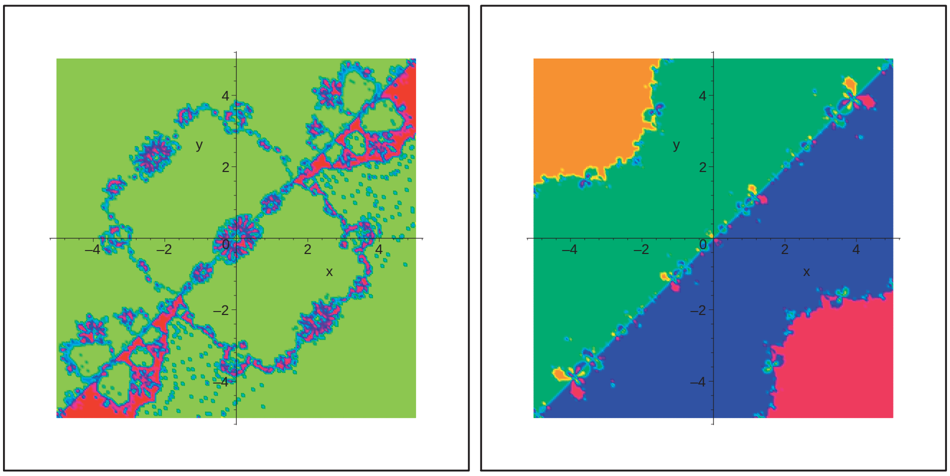


Figure 42: Neta's sixth order method (left) and Neta and Johnson's eighth order method (right)

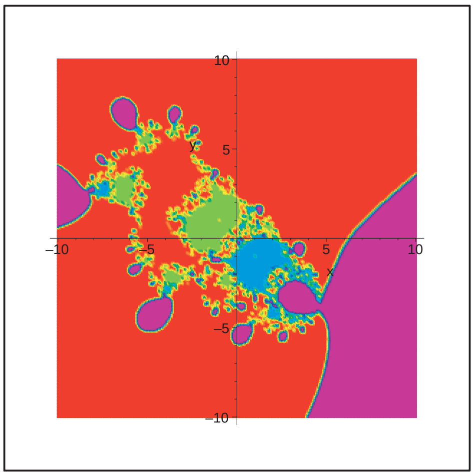


Figure 43: Neta and Petkovic's optimal eighth order (left) and Neta's sixteenth order method (right)

$$x^5 + 20ix^3 - 36x \tag{27}$$

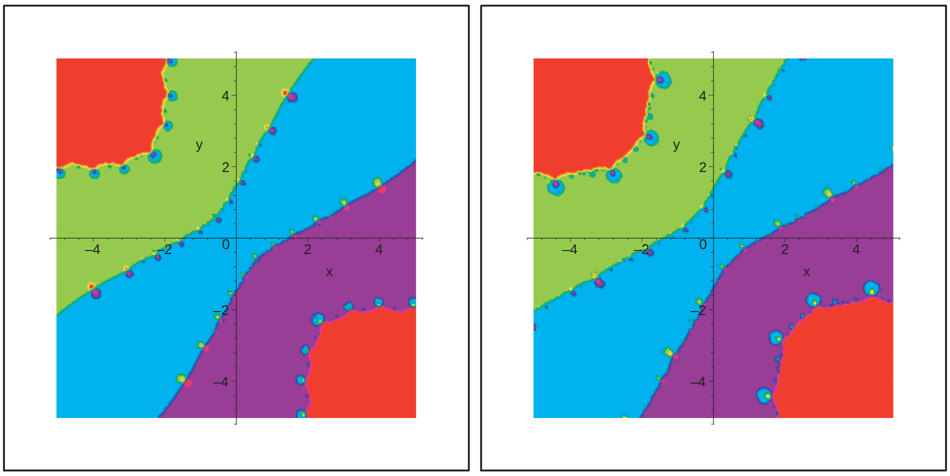


Figure 44: Newton's (left) and Halley's method (right) for the complex quintic polynomial whose roots are: -3+3i, 1-i, -1+i, 3-3i, 0

As can be seen, the sixth order and the optimal sixteenth order methods are not doing very well.

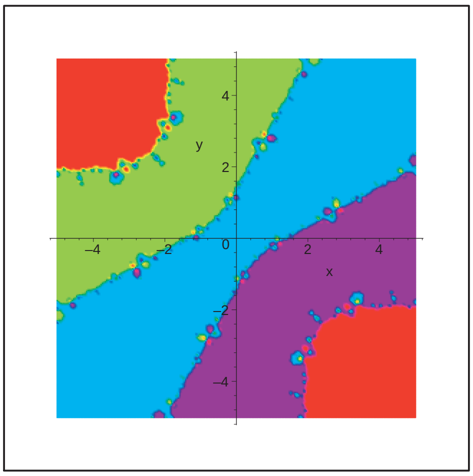


Figure 45: Kung and Traub's fourth order method (left) and Murakami's fifth order method (right)

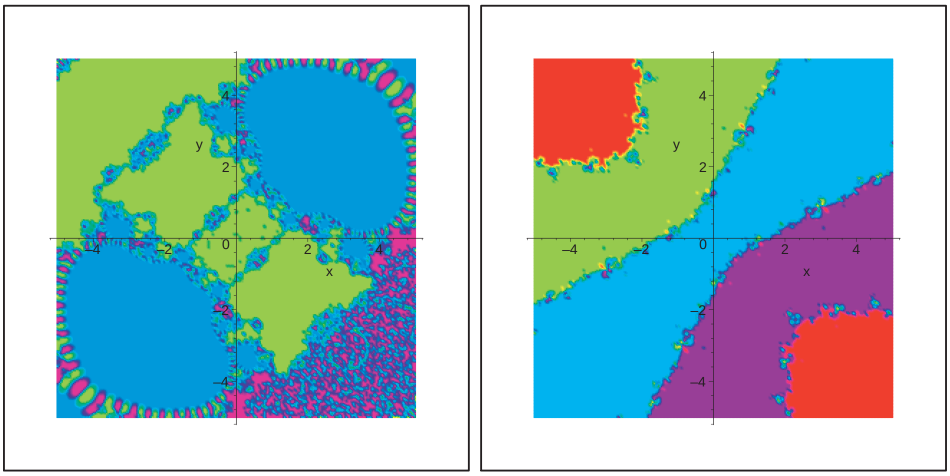


Figure 46: Neta's sixth order method (left) and Neta and Johnson's eighth order method (right)

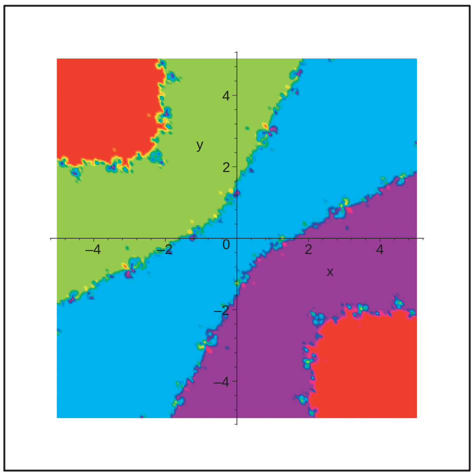


Figure 47: Neta and Petkovic's optimal eighth order (left) and Neta's sixteenth order method (right)

$$x^5 + (5-i)x^4 + (6-38i)x^3 + (-336+96i)x^2 - (3072-1536i)x - 10240 + 5120i \quad (28)$$

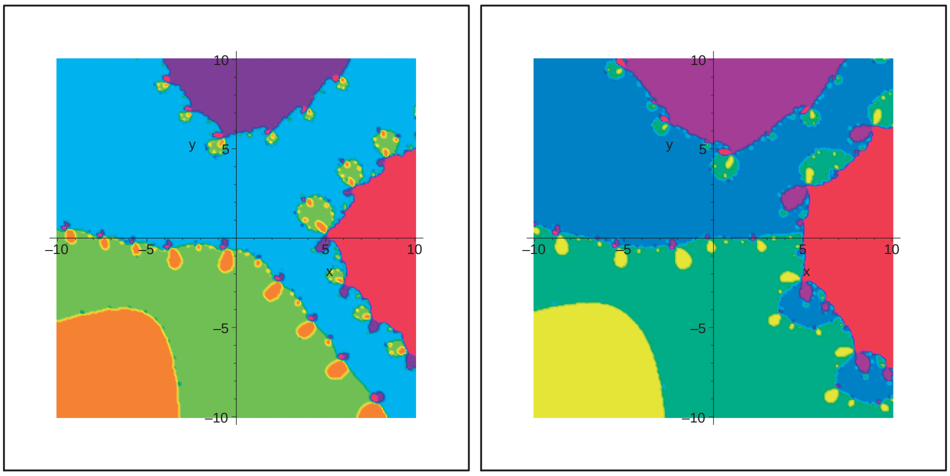


Figure 48: Newton's (left) and Halley's method (right) for the complex quintic polynomial whose roots are: -4+2i, -5-5i, -4-4i, 8i, 8

As can be seen, the optimal eighth and sixteenth order methods as well as the sixth order are not doing very well.

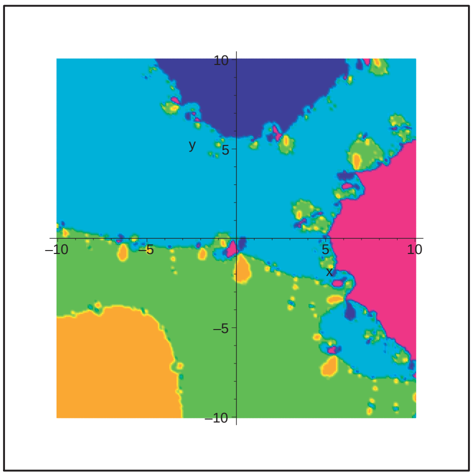
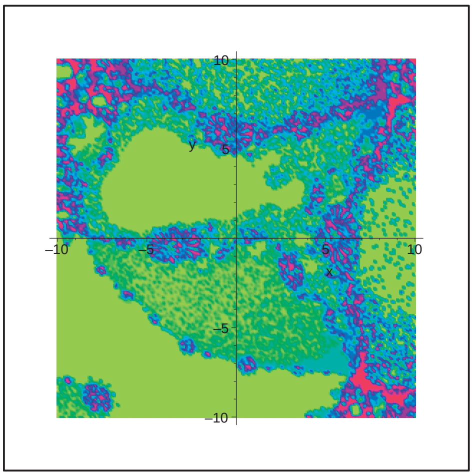


Figure 49: Kung and Traub's fourth order method (left) and Murakami's fifth order method (right)



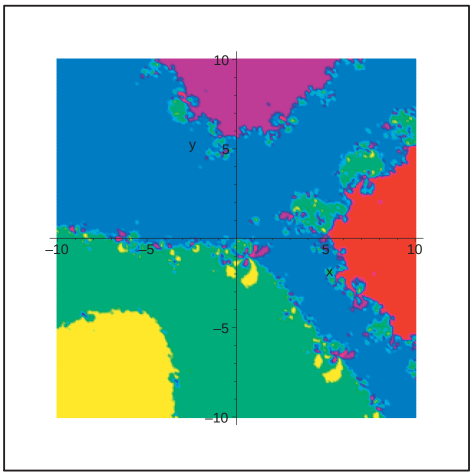


Figure 50: Neta's sixth order method (left) and Neta and Johnson's eighth order method (right)

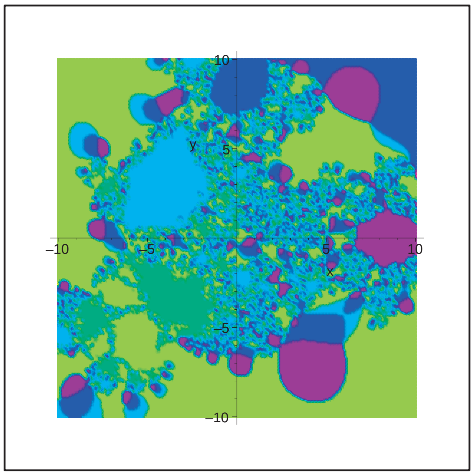


Figure 51: Neta and Petkovic's optimal eighth order (left) and Neta's sixteenth order method (right)

$$x^6-(1-2i)x^5+(6-39i)x^4-(219+2i)x^3-(1295-263i)x^2-(28428-9936i)x+8640+28512i \tag{29}$$

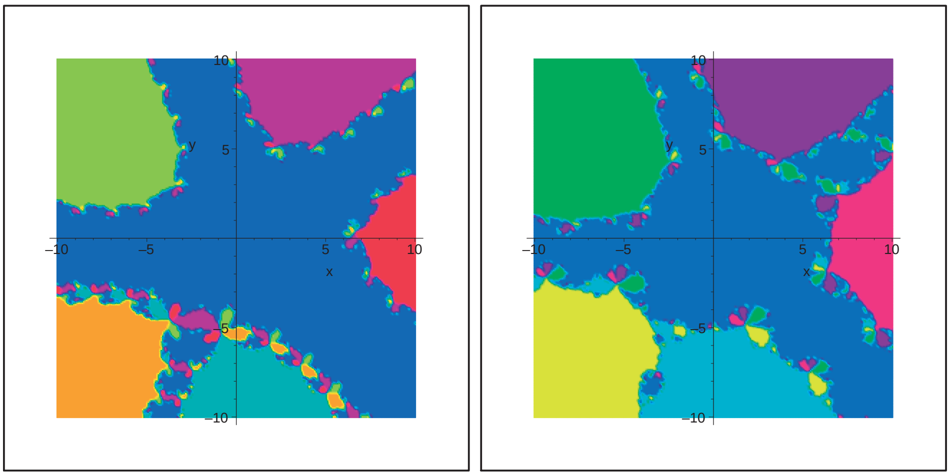


Figure 52: Newton's (left) and Halley's method (right) for the complex sixth degree polynomial whose roots are: 3+7i, -8i, -6-6i, i, -5+4i, 9

As can be seen, the optimal eighth and sixteenth order methods as well as the sixth order are not doing very well.

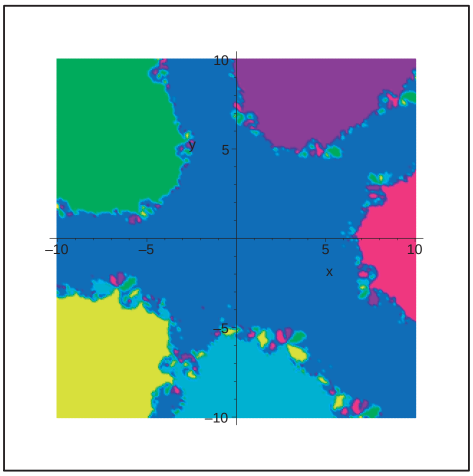
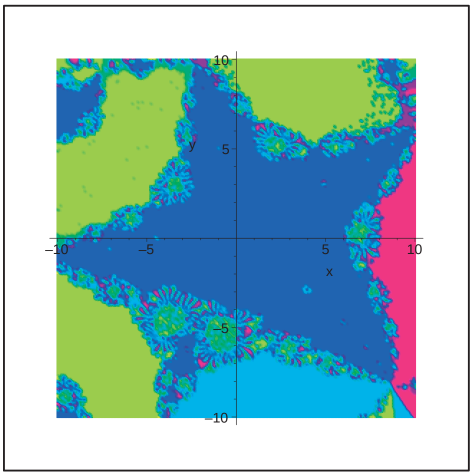


Figure 53: Kung and Traub's fourth order method (left) and Murakami's fifth order method (right)



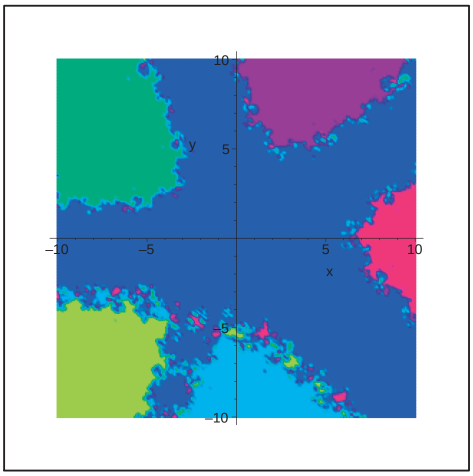


Figure 54: Neta's sixth order method (left) and Neta and Johnson's eighth order method (right)

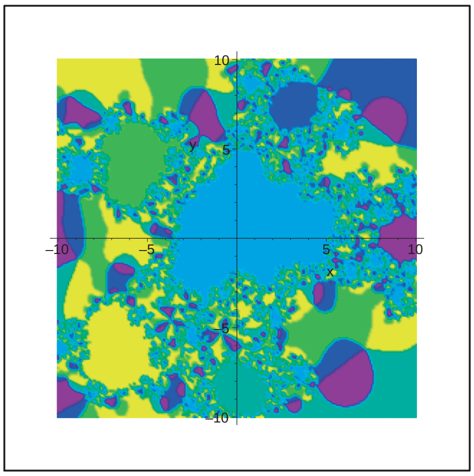


Figure 55: Neta and Petkovic's optimal eighth order (left) and Neta's sixteenth order method (right)

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